mble Ladies' College
Mathematics Department

Name: ______________________

SCEGGS Darlinghurst

HSC ASSESSMENT 2 21 May 2001

Mathematics

Year 12

Extension 1

TIME ALLOWED: $1\frac{1}{2}$ HOURS

Topics: ***Permutations and Combinations, Inverse Functions, Further Trigonometry, Binomial Probability.***

Percentage of final assessment: 35%

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- Ensure that your name is on this booklet.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Approved calculators should be used. Mathematical templates and geometrical instruments may be used.
- Begin each question on a new page - write your name at the top of each page.
- Hand up the question sheet as well as your answer pages.

QUESTION 1 **(18 Marks)** **Marks**

a) Evaluate $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ 3

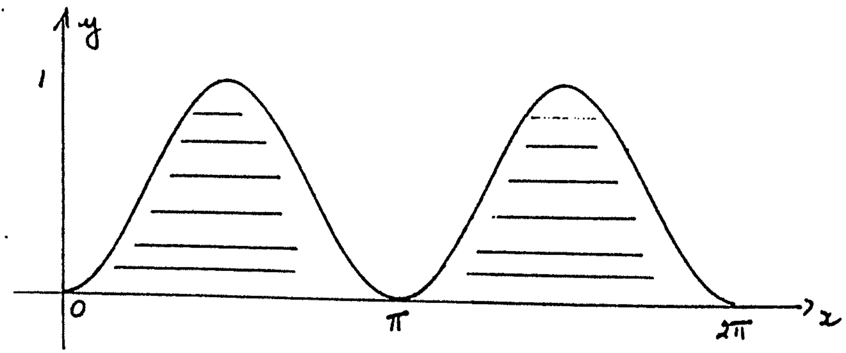
b) The probability that a machine fails in repeated trials is $0 \cdot 05$. If the machine is operated 10 times, find the probability that:

(i) it does not fail. 1

(ii) it fails more than once 2

(iii) the company that makes the machine states that "the machine has a 99% certainty that it will not fail more than once in 100 starts." How accurate is this statement? Explain your answer. 2

c)



The graph shown above is $y = \sin^2 x$ for $0 \le x \le 2\pi$ 3

Find the shaded area.

Marks

d.) There are six children in a group, sitting in a circle.

(i) In how many ways can they sit if they may sit anywhere? 1

(ii) In how many ways can they sit if two particular students are not allowed to sit together? 2

(iii) Three of the children are sisters. Find the probability that they sit together. 2

(iv) If the group contains three boys and three girls, find the probability that the boys and girls are seated alternately. 2

QUESTION 2 **(20 Marks)**

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a.) (i) Prove that $\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$ 1

(ii) Hence find the volume formed when the region contained by the x axis and the curve $y = \cos 2x$ between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ is rotated about the x axis. (Answer in exact form.) 4

b.) A tin contains a large number of red and green lollies mixed together in the ratio $3:2$. A handful of 20 lollies is taken from the tin.
Find the probability that:

(i) there are no red lollies in the handful. 1

(ii) there are 7 red lollies in the handful. 1

(iii) there are at least 18 red lollies. 2

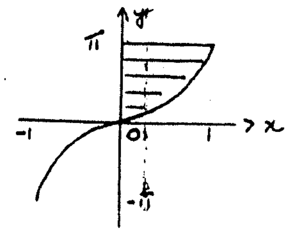
(iv) the handful is such that it can be shared between 5 children so that each child receives the same number of red lollies. (There is no need to calculate this answer.) 3

Marks

c.) Find the exact value of:

$$\sin\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{-5}{12}\right)\right)$$ 3

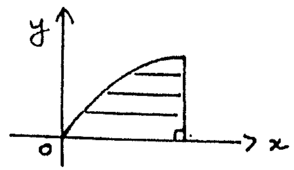
d.)



The curve shown is:
$y = 2\sin^{-1} x$

A region is shaded

(i)



The curve shown is a sine curve.
The region shaded has the same area as that shown above. 2

State the equation of the curve and the domain and range of the shaded region.

(ii) Hence find the volume formed when $y = 2\sin^{-1} x$ between $y = 0$ and $y = \pi$ is rotated about the y axis. 3

QUESTION 3 **(17 Marks)**

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a) (i) In how many ways can a committee of 6 people be formed from 5 women and 4 men? 1

(ii) How many committees are possible if a particular man and a particular woman refuse to serve together? 2

(iii) If one of the committees is chosen at random, find the probability that it contains more than 3 women. 3

b) (i) How many different arrangements are possible of the word **DELETED**? 1

(ii) In how many of these arrangements are the E's together? 2

(iii) If six of the letters are selected at random, how many different arrangements are now possible? 3

c) 4 married couples play 2 games of tennis doubles at the same time. How many ways can the pairings be arranged:

(i) if there are no restrictions? 3

(ii) the married couples play together? 2

QUESTION 4 **(17 Marks)** **Marks**

START A NEW PAGE

a) Consider the function $y = \frac{1}{2}\cos^{-1}\frac{x}{2}$

(i) State the domain and range of the function. 2

(ii) Sketch the function. 2

b) Evaluate $\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{dx}{4+9x^2}$ 4

c) (i) Sketch $f(x) = x^2 + 2x + 1$ 1

(ii) State a suitable domain (containing $x = 0$) for $f(x)$ such that $f(x)$ has an inverse function. 1

(iii) Sketch the inverse function $f^{-1}(x)$ showing all important features. 2

(iv) Find $f^{-1}(x)$

d) Consider $y = \cos^{-1}(\sin x)$

(i) Find $\frac{dy}{dx}$ 1

(ii) State an interesting fact about $\frac{dy}{dx}$ 1

(iii) Sketch $y = \cos^{-1}(\sin x)$ 3

END OF ASSESSMENT

Assessment Test Answers Extension 1 Test 2.

1) a) $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_1^{\sqrt{3}}$

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3} - \frac{\pi}{6}$

$= \frac{\pi}{6}$

b) (i) $(0.95)^{10} = 0.5987\ldots$

(ii) Prob(0, 1 failures)

$= (0.95)^{10} + 10\times(0.95)^9\times(0.05)$

$= 0.914\ldots$

∴ Prob(more than 1 failure)

$= 1 - 0.914\ldots$

$= 0.086\ldots$

(iii) Prob(not failing more than one)

= 100 successes + 99 successes

$= (0.95)^{100} + 100\times(0.95)^{99}\times(0.05)$

$= 0.037\ldots$

statement is extremely inaccurate.

c) Area $= 2\int_0^{\pi} \sin^2 x\,dx$

$= 2\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx$

$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$

$= \pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin 0}{2}$

$= \pi\ u^2$

d) (i) $5! = 120$

(ii) if together $= 4!\times 2 = 48$

if not together $= 120 - 48 = 72$.

(iii) if together $= 3!\times 3! = 36$

∴ probability $= \frac{36}{\ } = \frac{3}{\ }$

(iv) if alternating : $3!\times 2! = 12$

∴ probability $= \frac{12}{120} = \frac{1}{10}$

2) a)(i) $\cos 4x = \cos^2 2x - \sin^2 2x$

$= 2\cos^2 2x - 1$

∴ $\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$

(ii) $V = \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2 2x\,dx$

$= \frac{\pi}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos 4x + 1\,.dx$

$= \frac{\pi}{2}\left[\frac{\sin 4x}{4} + x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$= \frac{\pi}{2}\left(\frac{\sin 3\pi}{4} + \frac{3\pi}{4} - \frac{\sin\pi}{4} - \frac{\pi}{4}\right)$

$V = \frac{\pi^2}{4}\ u^3$

b) (i) $P(\text{no red}) = \left(\frac{2}{5}\right)^{20} = 1.0995\times 10^{-8}$

(ii) $P(7\text{ red}) = \binom{20}{7}\left(\frac{3}{5}\right)^7\left(\frac{2}{5}\right)^{13} = 0.015$

(iii) $P(18, 19, 20\text{ red})$

$= \binom{20}{18}\left(\frac{3}{5}\right)^{18}\left(\frac{2}{5}\right)^2 + \binom{20}{19}\left(\frac{3}{5}\right)^{19}\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^{20}$

$= 3.6\times 10^{-3}$ (approx)

(iv) The number of red lollies must be divisible by 5 i.e. 0, 5, 10, 15, 20

$\left(\frac{2}{5}\right)^{20} + \binom{20}{5}\left(\frac{2}{5}\right)^{15}\left(\frac{3}{5}\right)^5 + \binom{20}{10}\left(\frac{2}{5}\right)^{10}\left(\frac{3}{5}\right)^{10}$

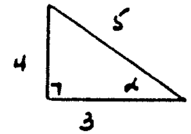
$+ \binom{20}{15}\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^{15} + \left(\frac{3}{5}\right)^{20}$

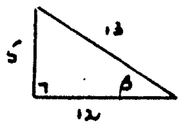
c) Let $\sin^{-1}\left(\frac{4}{5}\right) = \alpha$ $\left(\sin\alpha = \frac{4}{5}\right)$

$\tan^{-1}\left(\frac{-5}{12}\right) = \beta$ $\left(\tan\beta = \frac{-5}{12}\right)$

note that α is 1st quadrant

β is 4th quadrant





$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{4}{5}\times\frac{12}{13} - \frac{3}{5}\times -\frac{5}{13}$

$= \frac{63}{65}$

d) (i) D: $0 \le x \le \pi$

R: $0 \le y \le 1$

equation $y = \sin\frac{x}{2}$

(ii) $V = \pi\int_0^{\pi} \sin^2\frac{x}{2}\,dx$

$= \frac{\pi}{2}\int_0^{\pi} 1 - \cos x\,.dx$

$= \frac{\pi}{2}\left[x - \sin x\right]_0^{\pi}$

$= \frac{\pi}{2}\left[\pi - \sin\pi - 0 + \sin 0\right]$

$V = \frac{\pi^2}{2}\ u^3$

3) a) (i) $\binom{9}{6} = 84$

(ii) if included $= \binom{7}{4} = 35$

∴ if not together $= 84 - 35 =$

(iii) 4 women $= \binom{5}{4}\times\binom{4}{2} = 30$

5 women $= \binom{5}{5}\times\binom{4}{1} = 4$

∴ Probability $= \frac{34}{84} = \frac{17}{42}$

b) (i) $\frac{7!}{2!\times 3!} = 420$.

(ii) $\frac{5!}{2!} = 60$

(iii) E excluded $= \frac{6!}{2!\times 2!} = 180$.

D excluded $= \frac{6!}{3!} = 120$

L excluded $= \frac{6!}{3!\times 2!} = 60$

T excluded $= \frac{6!}{3!\times 2!} = 60$

Total number = 420

c) (i) there are 2 methods

either $\binom{8}{4}\div 2$ to select 4 peop

then $\binom{4}{2}\div 2$ to organize the four into doubles pairings.

∴ $\binom{8}{4}\times\binom{4}{2}\times\binom{4}{2}\div 8 = 315$

or $\binom{8}{2}\times\binom{6}{2}\times\binom{4}{2}\div 8 = 315$

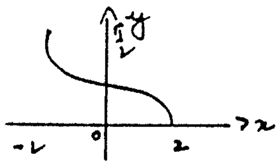
(ii) $\binom{4}{2}\div 2 = 3$

The division by 2 is the resul

of the games being played togethe

4) a)(i) D : $-2 \le x \le 2$

R : $0 \le y \le \frac{\pi}{2}$.



b) $\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{dx}{4+9x^2} = \frac{1}{9}\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{dx}{\frac{4}{9}+x^2}$

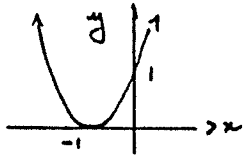
$= \frac{1}{9} \times \frac{3}{2}\left[\tan^{-1}\frac{3x}{2}\right]_{-\frac{2}{3}}^{\frac{2}{3}}$

$= \frac{1}{6}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$

$= \frac{1}{6}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$

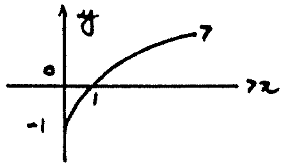
$= \frac{\pi}{12}$

c) (i)



(ii) $x \ge -1$

(iii)



(iv) $x = (y+1)^2$

$y+1 = \pm\sqrt{x}$

$y = -1 \pm \sqrt{x}$

$f^{-1}(x) = -1 + \sqrt{x}.$

d) $y = \cos^{-1}(\sin x)$

(i) $\frac{dy}{dx} = \frac{-1}{\sqrt{1-\sin^2 x}} \times \cos x$

$= \frac{-\cos x}{\sqrt{\cos^2 x}}$

(ii)

now $\sqrt{\cos^2 x} \ge 0$ by the definition of $\sqrt{\ }$ but $\cos x$ can be +ve or -ve depending on the value of x

$\therefore \frac{dy}{dx}$ can be 1 or -1

(iii) if $y = \cos^{-1}(\sin x)$

$-1 \le \sin x \le$ from the domain of an inverse cos. but this is true for all x

$\therefore$ domain of $\cos^{-1}(\sin x)$ is all real x.

the range is $0 \le y \le \pi$

Testing values gives the graph: